Examinations in the Context of Curriculum Content: Case Study of a 1926 Irish Mathematics Exam Paper

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Abstract

Examinations are used worldwide as a method of assessing student ability and learning. The content and form of these examinations has developed over the years as university entry and higher education has become mainstream. In this work, we collected a selection of mathematics exam papers dating from 1926 to 2020. While we may be tempted to use this collection to determine the capabilities and complexity required by the mathematical content covered at the time, we demonstrate that examination papers alone cannot provide a clear insight into depth of learning. We select two questions from a 1926 exam paper and demonstrate that a targeted and potentially rote-learned curriculum can mean that seemingly difficult topics can be reduced to repetition exercises, potentially without analysis or comprehension.

Keywords

Education, Curriculum, Examination, Mathematics

1. Introduction

In our current technological world, we seek the skills of mathematicians and problem solvers in our economy and society. However, a reoccurring narrative is that the material covered in state-run examinations is becoming easier and tending away from in-depth pure mathematical explorations [1]. Developments in recent Irish examinations have sought to focus on students' understanding and comprehension of content, and therefore has been criticized for "simplifying the curriculum" [2].

In this paper, we investigate the content of a 1926 exam paper and uncover the context behind the seemingly complex mathematical problems by revealing the targeted and superficial curriculum content. We demonstrate that examination papers cannot be used as a means of assessing the depth of knowledge expected of students unless they are considered in the context of the curriculum content.

We begin in Section 1.1 with a sample of the related work in this field. Following this, Section 1.2 introduces the Irish examination structure and the taxonomy of Irish exam papers we collected. In Section 2, we introduce the exam paper questions we have chosen to focus on in this study. In Section 3, we present the modern solutions to these two exam questions. Then, in Section 4 we leverage contemporary textbooks from the time period to investigate how stu-

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dents might have been expected to answer the same questions. We observe that the textbooks of the time provide only segmented explanation, and include examples closely matched to the examination questions. Section 5 reviews our findings and discusses the impact they have on our understanding of the content and examinations of the past. Finally, we conclude in Section 6.

1.1. Related work

Many researchers have studied exam papers to determine the depth of understanding expected of students and to evaluate the complexity of the topic under examination. In 2017, Mac an Bhaird et al. [3] studied undergraduate calculus assessments including assignments and examinations to determine the amount of critical reasoning versus imitative reasoning (rote learning and repetition) opportunities in assessments for students. In 2020, Akçay et al. [4] studied Turkish exam papers for 5th, 6th, 7th and 8th classes in second level education. The exam papers were examined with respect to the number and type of questions, the language used in the questions, the visuals used, and the cognitive level (according to Bloom's taxonomy [5]) of the questions. They show that the questions are mostly at the comprehension level, and critical reasoning questions are rare. Cebesoy et al. [6] studied a collection of science examination papers with the goal of investigating the types of questions used, the consistency between questions and objectives in the unit, and whether there included questions related to mathematics. With knowledge of the unit content and the exam papers, all these questions could be answered and they found that most question were simple multiple choice but that they did attempt to test students' mathematical knowledge. Kang and Saeed [7] evaluated 2014-2015 mathematical examination papers from the Secondary School Certificate (SSC) Examinations and from the General Certificate of Education (GCE), which run in parallel in Pakistan. Their study showed that the SSC questions are highly focused on factual knowledge and routine procedures. They conclude that these items assess knowledge of facts and algorithms only, and do not measure essential mathematical skills.

Exam papers are a valuable source of information to assessors of current mathematical programs. They are also useful as a method to allow us to evaluate the historical development of educational priorities and gain insights into examinations from the past. However, as we will show in this paper, sometimes exam papers alone cannot reveal whether students are expected to employ critical or imitative reasoning unless we also look at the content these examinations are based on. We chose two Irish past examination questions, which on the surface seem to require critical thinking and complex algorithmic reasoning, but once taken in the context of the textbooks of the day, reveal that only superficial retention was expected.

1.2. Exam paper collection

The Irish secondary school system, broadly speaking, currently examines students after three years of secondary education at an age of approximately 15–16 (the *Junior Certificate*, formerly the *Intermediate Certificate*) and again, after a further 2–3 years at the *Leaving Certificate*. The system has run since 1925, with disruption in only exceptional circumstances, such as the 2020–2021 Covid pandemic. However, it is worth noting that there has been significant changes over the years to the surrounding school system, such as the introduction of free secondary school education for all students in 1967.

In 2016, we began a collection of STEM (Science, Technology, Engineering and Mathematics) examination papers from the Irish second-level school system [8]. This collection was motivated by the sometimes imperfect memories of the syllabus and examinations that people had reported they had been through. As no collection of these examination papers was available to the public, or researchers, we aimed to construct an archive that would be of use to both the general public and researchers in STEM education.

A cross-section of the available Junior certificate and Leaving certificate subjects are included in this archive. At Junior Cert level this includes Maths, Science and Mechanical Drawing/Technical Graphics. At Leaving Cert level it includes Maths, Applied Maths, Physics, Chemistry, Biology, Technical Drawing/Graphics and Computer Science. Some of these subjects have not been available for the full period of the examinations (e.g. Biology and Computer Science). The archive is now largely complete, and has been compiled with the help of libraries, government bodies and the general public.

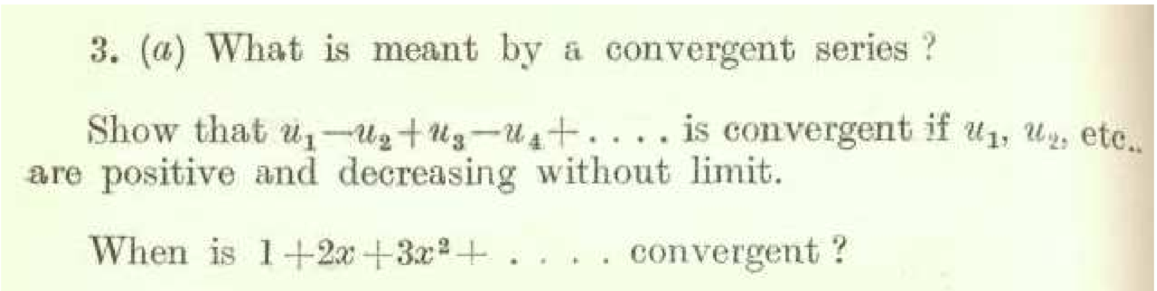
2. The questions

To examine a past examination paper, we chose the 1926 Leaving certificate mathematics paper [8]. This paper would be taken by students in their final examination in second level before potentially progressing to third level education. We select the 1926 examinations as these are some of the earliest examinations run by the newly established Irish state, and give us a good opportunity to view questions at a distance, where we have only limited preconceived knowledge of what students are being asked.

We have chosen two questions from the paper, Q3(a) and Q5, as these seem to differ the most from the mathematics exam questions that 21st century students would be expected to answer.

Q3: Convergence There are two parts to Q3, and you are asked to do either Q3(a) or Q3(b). We concentrate on the former, which is shown in Figure 2.

First, notice that the question includes some terminology that may not be familiar to modern readers. The phrase *decreases without limit* is used in a number of textbooks from the 1800s (including one of Agustus De Morgan's [9]), and appears to mean that the sequence decreases



3. (*a*) What is meant by a convergent series ?

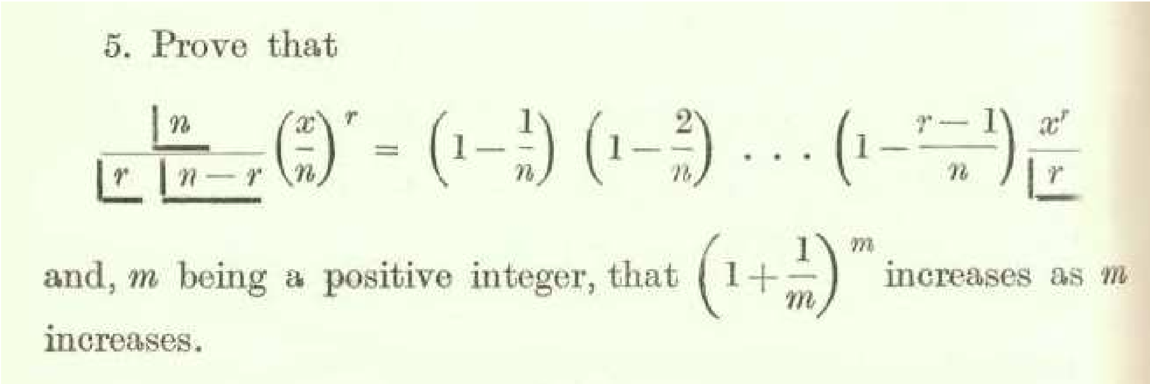
Show that $u_1 - u_2 + u_3 - u_4 + \ldots$ is convergent if u_1, u_2, etc., are positive and decreasing without limit.

When is $1 + 2x + 3x^2 + \ldots$ convergent ?

Figure 1: Leaving Certificate 1926 Higher Level Maths Q3(a).

to zero, without becoming constant. Thus, the question is asking about the Alternating Series Test [10].

Q5: Binomial Theorem Q5 is shown in Figure 2. It also contains some notation that could be unfamiliar to the modern reader $\underline{|n}$ is notation for $n!$ It concerns binomial coefficients and a well-known limit for e.



5. Prove that

$$\frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}\left(\frac{x}{n}\right)^r = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\frac{x^r}{\underline{|r}}$$

and, m being a positive integer, that $\left(1+\frac{1}{m}\right)^m$ increases as m increases.

Figure 2: Leaving Certificate 1926 Higher Level Maths Q5.

3. Possible Modern Solutions

In this section, we will provide modern solutions for the two questions in the 1926 exam paper.

Q3(a): Convergence Q3 appears to require definitions and proofs from an initial third-level course in real analysis. For example, one might first define convergence as

> A series with terms a_n converges if $\exists L \in \mathbb{R}$ so that $\forall \epsilon > 0$ we can find $N \in \mathbb{N}$ so that whenever $n \in N$ and $n > N$ we have
>
> $$\left|\sum_{k=1}^{n} a_k - L\right| < \epsilon.$$

The second part asks for a proof of the Alternating Series Test. We will not give the full details, but a typical proof of this might proceed as follows.

> Let s_n be the partial sums, then note that s_{2n} is an increasing sequence and s_{2n-1} is a decreasing sequence. A little effort will show that $s_2 \leq s_{2n} < s_{2n-1} \leq s_1$. So, we see that the subsequence of even terms is increasing and bounded above, and so by the Monotone Convergence Theorem is convergent. Similarly, the subsequence of odd terms is decreasing and bounded below, and also convergent.
>
> Now, armed with the convergence of these two subsequences, we observe that the difference of their limits is
>
> $$\lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1} = \lim_{n\to\infty} s_{2n} - s_{2n-1} = \lim_{n\to\infty} -u_{2n} = 0,$$

so the subsequences have the same limit. A short $\epsilon - \delta$ argument shows that a sequence composed of odd and even terms converging to the same limit converges to that limit.

The final part of the question can be addressed using the Ratio Test.

The Ratio Test shows that the given series will converge when

$$\lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right| = |x| < 1.$$

The same test tells us that the series will diverge if $|x| > 1$. The case when $x = 1$ is obviously divergent and $x = -1$ gives an alternating non-convergent sequence.

Thus we have answered Q3(a).

Q5: Binomial Theorem Q5 has two parts. In modern notation, the first part asks to show that

$$\frac{n!}{r!(n-r)!}\left(\frac{x}{n}\right)^r\frac{x^r}{r!} = \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{r-1}{n}\right)\frac{x^r}{r!}.$$

The solution can be given in a few lines of relatively straightforward algebra using factorials. One solution to the second part involves expanding $(1 + 1/m)^m$ using the Binomial Theorem and then observing that the terms of the binomial expansion can be rewritten using the first part of Q5 with $x = 1$ and $n = m$. The required result follows by noting that the terms on the right hand side decrease when $n = m$ increases, and that the binomial expansion for $n = m + 1$ has an extra positive term.

Thus these are what we would now consider to be correct solutions to the 1926 questions. Note, that the full derivation of these solutions might currently be considered outside the scope of a second-level mathematics course.

4. Solutions based on Contemporary Textbooks

We identified contemporary textbooks [11, 12] that use both the terminology of *decreasing without limit* and the older notation for factorials used in the exam paper. It seems reasonable that the students, or at least their instructors, must have been expected to have access to such a text. We aim to gain some insight into what might have been expected of a student in 1926 through these text books.

Q3(a): Convergence Beginning with Q3, for example, §123 of Hymers [11] provides the following definition for the convergence of a series:

> A series $u_1 + u_2 + u_3 + \ldots + u_n \ldots$ (to ∞) is called convergent, if the sum s_n of any number n of its terms approaches continually to a finite quantity s as its limit, when n is indefinitely increased; and divergent in the contrary case.

The word *continually* does not appear to be formally defined in the text, and thus the student does not have to struggle with any arguments involving ϵ.

In fact, this text provides a proof of the Alternating Series Test in §132, using almost the same notation and the same terminology as the Leaving Cert question.

> The alternating series $u_1 - u_2 + u_3 - u_4 +$ &c., is convergent, if the numerical value of the terms decreases without limit.
>
> For, by writing it in the forms $u_1 - (u_2 - u_3) - (u_4 - u_5) -$ &c. and $u_1 - u_2, +(u_3 - u_4) +$ &c.; we see that it is $> u_1 - u_2$, and $< u_1$,and therefore is convergent.

This argument misses some of the subtleties of the full proof, and never uses the requisite fact that the sequence goes to zero. However, we suspect that a proof along this line is expected, possibly having been memorised by the student.

The Ratio Test is covered in §125 of Hymers' text, and §129 gives a version of the Ratio Test that can easily be applied to the last part of Q3(a) when $|x| \neq 1$. The same section seems to skirt around the situation when $|x| = 1$ without saying exactly what happens in this case.

Q5: Binomial Theorem Hymers does not spend much time on Binomial Theorem, so for Q5, we turn to Atkins' textbook [12], which has a chapter devoted to the topic. This chapter also introduces the $|\underline{n}$ notation for factorial and uses the phrase *decreases without limit* in an example in the chapter. In the example, the student is asked to

> Find the limit of
> $$\left(1 + \frac{1}{x}\right)^x,$$
> when x increases without limit.

The solution shows that the terms can be expanded as

$$\left(1 + \frac{1}{x}\right)^x = 1 + \frac{1}{1} + \frac{1 - \frac{1}{x}}{1.2} + \frac{\left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{x}\right)}{1.2.3} + \text{\&c.}$$

Putting $x = m$ gives the terms in exactly the form shown in the Leaving Cert Q5, and allows us to show the sequence is increasing in m by considering each term as m increases.

5. Discussion

The above analysis gives us an insight into the mathematics examination of its day. It seems to be more reliant on rote learning than one might have expected. The expected depth of student understanding, given that the questions seem so close to exercises from the textbook, seems limited. Indeed, the textbooks do not cover convergence at a level that would allow a student to fully understand the questions asked. Secondly, if one wishes to make the argument that examinations were more difficult in the past, then this suggests caution in the inspection of exam papers without the context of the curriculum content and textbooks. In fact, we would propose that investigation of historic learning and comparisons against such should be ideally undertaken with the examination papers, the textbooks of the day, the curriculum, and the expected marking schemes for such examinations.

6. Conclusion

The study of past exam papers is beneficial for informing our future education methodologies and curriculum, and can help us to improve and rectify mistakes from the past. Given context, they are an excellent source of information and we encourage anyone interested to delve deeper into our full STEM Irish exam paper archive [8].

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